

## Problem Solving In Singapore Math

written by Andy Clark

Ever since Singapore scored first in both 4th and 8th grades in the Trends in International Math and Science Study (TIMSS) comparison assessments in 1995, 1999, and 2003, and in the top three in 2007, math educators have been interested in the secret of Singapore's success. While many factors have been catalogued – a coherent national curriculum, teacher training, a public belief in the importance of math to the national economy – without fail, all descriptions emphasize the importance of problem solving in the Singapore curriculum. <sup>1</sup>(Leinwand 2008, AIR 2005, Lianghuo Fan & Yan Zhu 2007) This paper will consider how problem solving is taught in Singapore and why it has proven so successful.

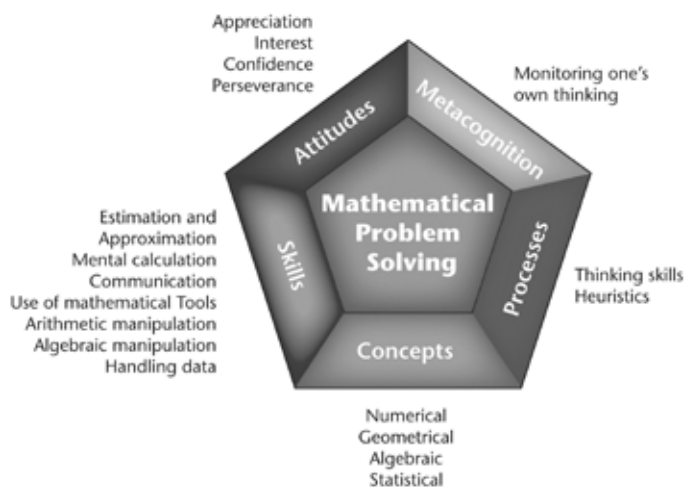
### A Greater Emphasis on Problem Solving

Interestingly, the first Singapore math curriculum, which was written in the 1980s, did not emphasize problem solving. It was not until 1991, and the writing of a new curriculum in 1992 that Singapore began emphasizing problem solving in its curriculum. As described on the Singapore Ministry of Education's Web site, problem solving is now the primary goal:

*Mathematical problem solving is central to mathematics learning. It involves the acquisition and application of mathematics concepts and skills in a wide range of situations, including non-routine, open-ended and real-world problems.*

*The development of mathematical problem solving ability is dependent on five interrelated components, namely, Concepts, Skills, Processes, Attitudes and Metacognition. (CPPD 2006)*

#### Singapore's Mathematics Framework



The Singapore Ministry of Education uses a graphic to represent their vision for mathematics teaching: a pentagon, with problem solving in the center and these five interdependent, necessary elements surrounding it. Textbooks, written specifically to address this structure, provide constant support for understanding all five elements. Students are encouraged to consider how they think, how they communicate, and how they solve problems, so they can apply their skills to subsequent problems. In its latest efforts, the Ministry is working to increase student communication skills and metacognition during problem solving.

If this sounds familiar to educators in the United States, it should. While math education in the 1970's emphasized the importance of basic skills, in reaction to the "new math" of the fifties and sixties, the 1980's and 1990's saw the emergence of problem solving in national and state standards. In the 1980 *Agenda for Action*, and the 1989 and 2000 versions of *Principles and Standards*, the National Council of Teachers of Mathematics (NCTM) has similarly identified the centrality of problem solving to the math curriculum. (NCTM 1989, 2000) Most U.S. textbooks now include "problem solving strategies and problems." Additionally, American researchers have conducted many studies on how to develop problem-solving abilities. (Schoenfeld 1985, 1992; Krulik and Rednick 1987; Qin, Johnson, Johnson 1995)

### **Differences in Performance Levels**

Despite the increased emphasis on problem solving in the United States, students in Singapore continue to perform better in math. I would suggest there are five major reasons for this difference in performance:

1. Problem solving is embedded in Singapore texts, not as a separate activity but as central to every skill and concept discussion.
2. The problems that Singapore students work on are much more complex than those in typical American texts. Two- and three-step problems are the norm.
3. Non-routine, as well as routine problems are included in every grade level.
4. Students are taught specific problem-solving strategies in a carefully sequenced manner beginning in second grade. The most famous of these—model drawing—is used to solve word problems initially, but once acquired as a skill, it becomes useful for solving non-routine problems as well.
5. Student attitudes are addressed and supported.

Lets take a closer look at each of these critical differences.

### **Embedded Problem Solving**

Each time a new concept is introduced in the Singapore textbooks, problem solving is central to the discussion. Consider the concepts of perimeter and area. Just as they do in the U.S., third graders in Singapore learn about these concepts. But in Singapore, students immediately consider them from several perspectives. So rather than just calculating the perimeter of a square with side length 3, students are asked:

*A square of side 3 cm is formed using a piece of wire. The wire is straightened and then bent to form a triangle with equal sides. What is the length of each side of the triangle? (MIF 2nd grade)*

When learning about area of rectangles, students don't just calculate height times width, but are asked to find the side length of a square that has the same area as an 8 x 2 rectangle. In other words, whenever a skill or concept is taught, it is also applied in a problem setting. The problem may not even look like other problems that are modeled. Students learn that a single skill can be applied to a wide range of problems.

## Complex Word Problems

Complex word problems are the norm in the Singapore textbooks. Yet students begin with very simple problems.

*There are 5 boxes of pencils. Each box has 12 pencils. How many pencils altogether?*

By the end of the third grade chapter, however, they solve problems such as this one:

*Shawn and Trish scored 36 goals in all. Shawn scored three times as many goals as Trish. How many goals did Trish score?*

or

*Flo saves 4 times as much money as Larry. Maria saves \$12 less than Flo. Larry saves \$32. How much does Maria save?*

In other words, the Singapore textbooks take seriously the proposition that the purpose of math is to solve complex problems. Multiple-step word problems are introduced in the primary grades and become increasingly challenging in the higher grades. Thoughtfulness is displayed, though, in the simplicity of the initial problems. As students develop confidence, they tackle more complicated problems, including non-routine ones.

## Solving Non-routine Problems and the Use of Heuristics

In addition to complex word problems, Singapore curriculum emphasizes non-routine problems—those that go beyond the application of specific computation. Solutions to such problems often require a number of different strategies, or heuristics. Before looking at specific problems from the curriculum, let's review the history of non-routine problems.

The NCTM (2000) defines problem solving as “engaging in a task for which the solution method is not known in advance” (p. 52). Johnson, Herr, Kysh (2004) define problem solving as “knowing what to do when you don't know what to do” (p. 3). But perhaps the best definition may come from the math teacher who did the most in the last century to include problem solving in the math curriculum, George Polya. Polya (1965) says, “solving a problem means finding a way out of a difficulty, a way around an obstacle, attaining an aim which is not immediately attainable.”

Polya went on to describe problems, now called non-routine problems, which did not lend themselves to mechanical application of one or more straightforward algorithms. These are problems in which you have to figure out *how* to proceed, as well as to calculate successfully.

The Singapore textbook writers are very familiar with Polya's work and include it in manuals designed for teachers. (Huat & Huat 2006) In particular they cite the 4-step model that Polya recommended: understand the problem, devise a plan, carry out the plan, and reflect on the solution. While this same structure is in many American textbooks as well, it is the special attention that Singapore math pays to steps 1 and 4 that may set it apart. In the first step, students connect a problem to prior problems. In the reflection stage students consider *Would alternate solutions take a shorter time? Can the method be applied to other problems?* (Huat 2006)

The Singapore curriculum and textbooks build on the work of Polya by teaching specific strategies for problem solving, what Polya called heuristics. Many of these—such as “look for a pattern,” “draw a picture,” “simplify the problem”, and “work backwards”, are included in American texts. But too often, American texts teach these strategies by “type.” That is, students learn “look for a pattern” and then they are presented with problems that can be solved in that way. In Singapore, students are instead encouraged to consider which strategy will work best for a particular problem. They are introduced to the strategies or heuristics and then given a variety of non-routine problems to solve.

Let’s look now at a third grade example.

*Mr. King has a total of 19 geese, chickens and ducks on his farm. He has 3 more chickens than geese. He has 2 fewer ducks than geese. How many ducks does he have?*

Try to solve this. Then imagine a third grade student solving it. Singapore students are taught to draw models to represent the situation. Students might use the most basic strategy of guess and check or they may try it more systematically. For instance, one student took 19 counters, assigning 1 to the duck 3 to the geese and 6 to the chickens. The student then added a counter to each row until all 19 had been used.

Other students used a “work backwards” strategy. They know that if they take away 5 chickens and 2 geese, then they will be left with three equal piles. Since there are 12 counters or animals left, this means there must be 4 of each, hence 4 ducks. Notice that many of these strategies lead to the generalization students will use later in algebra to solve the problem. Encouraging multiple approaches and evaluating their effectiveness is the essence of good problem solving.

## **Model Drawing**

The most famous and developed of the heuristics (strategies) taught in Singapore is “model drawing,” sometimes called “bar modeling” in the U.S. Beginning in 2nd grade, students are taught to use rectangular shapes to model a word problem. These models:

- help students visualize abstract math relationships through pictorial representations.
- use rectangular blocks because they are easily divided.
- can be used before students know algebraic solutions, and can be used to model algebraic relationships.

Dr. Yeap Ban Har of Singapore’s National Institute of Education describes the model method this way:

*The use of “model method” provides students with a means to handle information, deal with complexity, and at the same time, communicate their thinking through the use of visual models which they can manipulate.*

These models begin quite simply and typically model word problems. In the Grade 2 problem at right, a rectangle, divided into two parts, models a simple subtraction situation. There are 20 eggs, 7 are duck eggs. How many are chicken eggs? Other models include simple multiplication and division. By 5th grade, however these models increase significantly in complexity. In fact, model drawing proves to be a powerful tool for non-routine as well as routine problems. Consider the following problem:

*25% of the fish in a bowl are guppies.  
The same number of guppies as were originally in the bowl are then added.  
What percentage of the bowl is now guppies?*

Initially students are confused because there do not appear to be enough numbers. Yet a simple diagram can help. Students draw a rectangle, color one-fourth as guppies. Then they add a length to the rectangle that is the same size as the guppies portion. The picture shows clearly that 2 of the 5 parts of the bowl are guppies or 40%.

Careful attention to the teaching of heuristics, to moving from simple to complex problems, and to sequencing the problems in such a way as to move from routine to non-routine problems is a hallmark of the textbooks. Still, problem-solving does not account for all the differences in performance between students in Singapore and the US. Attitudes toward mathematics and the development of metacognitive skills also play a role.


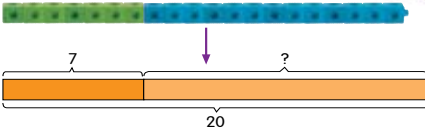
### Building Enthusiasm and Metacognition

In Singapore, efforts to develop positive attitudes and to improve metacognitive skills are evident in all aspects of mathematics learning, including classroom learning materials and the information sent home to students and families. Teachers are encouraged to solve mathematics problems themselves, so they develop a deeper understanding. Developing positive attitudes towards problem solving is a central focus— one that helps all students to feel better about math.

Ironically, at the same time as many states in the U.S. are depending exclusively on multiple choice tests, the Singapore assessment system is moving to incorporate more open ended assessment. Only 20% of the 6th grade placement test includes multiple-choice items. But Singapore’s educators believe that constructed response problems “require students to be able to read reasonably complex paragraphs. In addition students are expected to be able to communicate their thinking and methods by showing how they reason and arrive at the answer.” (Yeap 2008)

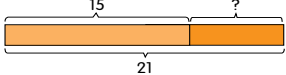
**Finding 1 set of objects from a group**

3 Ahmad bought 20 chicken eggs and duck eggs. There were 7 duck eggs. How many chicken eggs were there?

$20 - 7 = 13$   
There were 13 chicken eggs.

4 A school set up a new aquarium. There are 21 fishes in it. 15 fishes were given by the parents. The rest were bought by the school. How many fishes did the school buy?



-  =   
The school bought  fishes.

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My Pals Are Here: Maths,  
Grade 2, page 62

When addressing the winners of the high school math prizes, Ms Grace Fu, Senior Minister of State, Ministry of National Development and Ministry of Education Learning of Mathematics told the winners:

*Mathematics will imbue you with problem-solving skills such as the ability to understand a problem, identify the relevant information, look for relationships and patterns, make your own conjectures and apply mathematical knowledge and tools to solve or prove them. Mathematics, thus, provides good opportunities for the training of the mind to think critically and adapt to new situations, something that will be valuable to your future careers. (2008)*

In Singapore, a country lacking natural resources and half the size of New York City, human capital is the most precious resource. The Singapore curriculum and textbooks recognize that developing problem-solving skills and creativity is a requirement for the 21st century.

This belief, in addition to careful attention to teaching and cultivating problem-solving skills, are lessons worth considering as we try to make our students competitive in the global marketplace in which they live.



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